MICROCOPY RESOLUTION TEST CHART
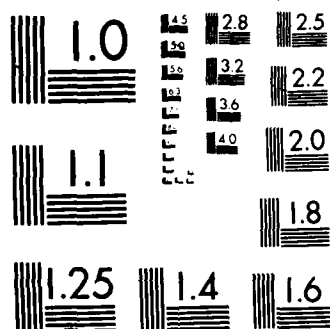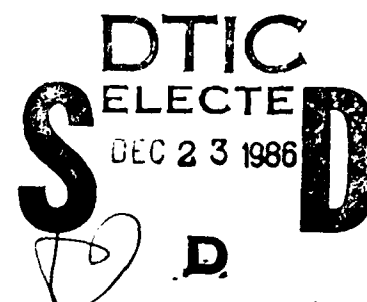
CYCLIC COLORATION OF 3-POLYTOPES

by

Michael D. Plummer*
Vanderbilt University
Nashville, Tennessee, USA

and

Bjarne Toft
Odense University
Odense, Denmark

1. Introduction and terminology

In this paper, all graphs will be finite, loopless and will have no parallel lines. Let G be a 2-connected planar graph with $|V(G)| = p$ points. Suppose G has some fixed imbedding $\phi : G \to \Re^2$ in the plane. The pair (G, ϕ) is often called a **plane graph**. A cyclic coloration of (G, ϕ) is an assignment of colors to the points of G such that for any face-bounding cycle F of (G, ϕ), the points of F have different colors. The **cyclic coloration number** $\chi_c((G, \phi))$ is the minimum number of colors in any cyclic coloration of (G, ϕ).

This concept was introduced in Ore and Plummer (1969) and was actually first formulated and considered by them in its planar dual form. That is to say, what is the smallest number of colors necessary to color the *faces* of a plane graph in such a way that no two faces with a common boundary *point* receive the same color. It was shown that if G is a 2-connected graph with imbedding ϕ in the plane, then $\chi_c(G, \phi) \leq 2\rho^*(G, \phi)$, where $\rho^*(G, \phi)$ represents the number of points in any longest face-bounding cycle in (G, ϕ). Let us henceforth denote the set of points in the cycle bounding face F by ∂F.

Recently, Borodin (1984, 1985) has announced an improvement of this upper bound to $2\rho^*(G, \phi) - 1$, although the proof is not presented in either paper.

The main result of the present paper is to show that if (G, ϕ) is a 3-connected plane graph, then $\chi_c(G, \phi) \leq \rho^*(G, \phi) + 9$. Moreover, we show that if ρ^* is sufficiently large or sufficiently small, then we can improve

---

this bound on χ_c somewhat.

Before proceeding further, several remarks are in order. When we speak of an *imbedding* ϕ of planar graph G in the plane, we mean a topological imbedding in the usual sense (see White (1973)) and would remind the reader that when G is 2-connected, every face is bounded by a cycle. Moreover, since parallel lines are not allowed in this paper, each such bounding cycle must have size at least three.

The main results in the present paper are concerned with 3-connected planar graphs. These are just the family of "3-polytopes" by a celebrated theorem of Steinitz (1922). (For more on graphs and polytopes the reader is referred to Grünbaum (1967).) We would also remind the reader that since we assume that our graphs are 3-connected, by yet another classical result, this one due to Whitney (1932), there is essentially only one way to imbed them in the plane. Thus from this point on when we refer to any 3-connected planar graph, we can ignore commenting upon which particular plane imbedding of the graph we are talking about. Accordingly, we will shorten the notation (G, ϕ) to simply G. Note that Tutte (1963) proved that the face-bounding cycles of a 3-connected planar graph are exactly the non-separating induced cycles of the graph.

For any terminology not defined in this paper, we refer the reader to Harary (1969) or to Lovász and Plummer (1986).

2. Some preliminary results

There are two principal tools we shall need to prove our main result. The first of these is the so-called *theory of Euler contributions* initiated by Lebesgue (1940) and further developed by Ore (1967) and by Ore and Plummer (1969). Let v be any point in a connected plane graph (G, ϕ). Define the **Euler contribution** of v, $\Phi(v)$, by

$$\Phi(v) = 1 - \frac{\deg v}{2} + \sum_{i=1}^{\deg v} \frac{1}{x_i},$$

where the sum runs over the face angles at point v and x_i denotes the size of the ith face at v.

We shall require several simple lemmas. (For proofs see Ore (1967) or Plummer (1985).) The first lemma is essentially due to Lebesgue (1940).

2.1. LEMMA. *If (G, ϕ) is a 2-connected plane graph, then $\sum_v \Phi(v) = 2$.* ∎

2.2. LEMMA. *Let (G, ϕ) be a 2-connected plane graph. Then for all $v \in V(G, \phi)$, $\Phi(v) \leq 1 - \deg v/6$.* ∎

It follows from Lemma 2.1 that there must exist a point v in any 2-connected plane graph (G, ϕ) with $\Phi(v) > 0$. Let us agree to call any such point $v \in V(G, \phi)$ a **control point**. (This terminology was first used in Plummer (1985), in which the author studied the extension of partial matchings in planar graphs.)

It is well-known, of course, that any plane graph has points v with $\deg v \leq 5$. We would like to emphasize, however, that Lemma 2.2 tells us that in any 2-connected plane graph we must have *control* points with degree ≤ 5. Moreover, for any control point v, we have the inequality

$$\sum_{i=1}^{\deg v} \frac{1}{x_i} > \frac{\deg v}{2} - 1. \tag{1}$$

Since we are assuming that G is 2-connected without parallel lines, each $x_i \geq 3$ and so inequality (1) yields the following three diophantine inequalities:

$$\deg v = 3: \qquad \sum_{i=1}^{3} \frac{1}{x_i} > \frac{1}{2}$$

$$\deg v = 4: \qquad \sum_{i=1}^{4} \frac{1}{x_i} > 1$$

$$\deg v = 5: \qquad \sum_{i=1}^{5} \frac{1}{x_i} > \frac{3}{2}.$$

The solutions to these inequalities are listed next for convenience. (Note that for the sake of conciseness, we list each solution in monotone non-decreasing order, although other cyclic orderings of faces of these sizes about a point are certainly possible and must be considered. (See Ore and Plummer (1969).) The number A listed in each case is the total number of points different from v on the union of the cycles bounding the faces containing v.

4

$$\deg v = 3: \quad (3,3,x) \quad x = 3,\ldots \quad A = x$$

$\deg v = 3:$	$(3,3,x)$	$x = 3,\ldots$	$A = x$
	$(3,4,x)$	$x = 4,\ldots$	$A = 1 + x$
	$(3,\,^{\cdot},x)$	$x = 5,\ldots$	$A = 2 + x$
	$(3,6,x)$	$x = 6,\ldots$	$A = 3 + x$
	$(3,7,x)$	$x = 7,\ldots,41$	$A = 4 + x$
	$(3,8,x)$	$x = 8,\ldots,23$	$A = 5 + x$
	$(3,9,x)$	$x = 9,\ldots,17$	$A = 6 + x$
	$(3,10,x)$	$x = 10,\ldots,14$	$A = 7 + x$
	$(3,11,x)$	$x = 11,12,13$	$A = 8 + x$
	$(4,4,x)$	$x = 4,\ldots$	$A = 2 + x$
	$(4,5,x)$	$x = 5,\ldots,19$	$A = 3 + x$
	$(4,6,x)$	$x = 6,\ldots,11$	$A = 4 + x$
	$(4,7,x)$	$x = 7,8,9$	$A = 5 + x$
	$(5,5,x)$	$x = 5,\ldots,9$	$A = 4 + x$
	$(5,6,x)$	$x = 6,7$	$A = 5 + x$
$\deg v = 4:$	$(3,3,3,x)$	$x = 3,\ldots$	$A = 1 + x$
	$(3,3,4,x)$	$x = 4,\ldots,11$	$A = 2 + x$
	$(3,3,5,x)$	$x = 5,6,7$	$A = 3 + x$
	$(3,4,4,x)$	$x = 4,5$	$A = 3 + x$
$\deg v = 5:$	$(3,3,3,3,x)$	$x = 3,4,5.$	$A = 2 + x$

The second tool we require has to do with the concept of a *contractible line* in a 3-connected graph. A line $e = uv$ joining points u and v in a 3-connected graph G is said to be **contractible** if upon identifying the points u and v and deleting the loop uv and any parallel lines formed thereby, we have a graph which is again 3-connected. We denote the resulting graph by $G \circ e$ or $G \circ uv$. Thomassen (1980) proved that every 3-connected graph with at least five points has a contractible line. Recently, Ando, Enomoto and A. Saito (1985) have sharpened this result by showing that, in fact, any such graph must have at least $\lceil p/2 \rceil$ such lines. Actually, they obtain more precise information as to the location of these contractible lines and it is these results which we shall use in obtaining our main result. We state them here for the sake of completeness.

The first result is Theorem 2 of the paper of Ando, et al.

2.3. THEOREM. *Let G be a 3-connected graph with at least five points. Then each point of degree 3 is incident with a contractible line.* ∎

The second result is Corollary 5 of their paper.

2.4. THEOREM. *Let G be a 3-connected graph with at least five points. Suppose $v \in V(G)$ and $\deg v \geq 4$ and suppose further that no line incident with v is contractible. Then at least three neighbors of v have the properties that each is of degree 3 and each is incident with exactly two contractible lines.* ∎

3. The main results

We are now prepared to prove our main theorem.

3.1. THEOREM. *Let G be a 3-connected plane graph with maximum face size $\rho^*(G)$. Then $\chi_c(G) \leq \rho^*(G) + 9$.*

PROOF. The proof proceeds by induction on $p = |V(G)|$. Since G is 3-connected, $p \geq 4$. If $p = 4$, then $G = K_4$ and we have $\chi_c(K_4) = 4 < 12 = \rho^*(K_4) + 9$.

So suppose $p \geq 5$ and that the result holds for all 3-connected plane graphs having fewer than p points. Let G be any 3-connected plane graph with p points. The reader should note immediately that contracting any line in such a graph G cannot result in an increase in ρ^* nor can it destroy planarity.

Now by Lemmas 2.1 and 2.2, graph G has a control point v with $\deg v \leq 5$.

Suppose first that v is incident with a contractible line $e = uv$. Then by the induction hypothesis, graph $G \circ e$ has a cyclic coloration in $\leq \rho^*(G \circ e) + 9 \leq \rho^*(G) + 9$ colors. Such a cyclic coloration of $G \circ e$ induces a partial cyclic coloration of G in which only point v is not colored, where u gets the color of the new contracted point $[uv]$ and all other points keep their colors. Now appealing to the list of solutions presented in Section 2, we see that $(\bigcup \partial F_i) - v$, where the union is taken over all faces F_i around v, has $A \leq x + 8$ different points. Thus in the partial coloring of G we have used at most $x + 8 \leq \rho^*(G) + 8 < \rho^*(G) + 9$ colors on $(\bigcup \partial F_i) - v$. Thus we can extend the partial coloration to a valid cyclic coloration of G in no more than $\rho^*(G) + 9$ colors, as there are $\geq (\rho^*(G) + 9) - (x + 8) \geq 1$ colors available for v.

Suppose next that v is not incident with a contractible line. Then by Theorems 2.3 and 2.4, $\deg v$ is 4 or 5 and at least three neighbors of v have degree 3 and are incident with a contractible line. Let a be a neighbor of v of degree 3 incident with a contractible line e, where we may assume that a is not on the x-face. By the induction hypothesis, the graph $G \circ e$ has a cyclic coloration in $\leq \rho^*(G \circ e) + 9 \leq \rho^*(G) + 9$

colors, inducing a partial cyclic coloration of G in which only a is not colored. If the three faces F_1, F_2 and F_3 around a have sizes y_1, y_2 and y_3 respectively, then $(\partial F_1 \cup \partial F_2 \cup \partial F_3) - a$ has at most $y_1 + y_2 + y_3 - 6$ points.

Relabeling faces F_1, F_2 and F_3 if necessary, we may suppose that F_1 and F_2 are faces at v. Then we may assume from the listing in Section 2 of the five possible cases that $y_1 + y_2 \leq 8$. Hence the partial cyclic coloration may be extended to a valid cyclic coloration of G in no more than $\rho^*(G) + 9$ colors, as there are $\geq (\rho^*(G) + 9) - (y_1 + y_2 + y_3 - 6) \geq (\rho^*(G) + 9) - (y_3 + 2) \geq 7$ colors available for point a. ∎

In the next theorem we show that if $\rho^*(G)$ is sufficiently *small*, we can improve the bound of Theorem 3.1.

3.2. THEOREM. *Let G be a 3-connected plane graph. Then:*

$$
\begin{array}{lll}
(i) & \rho^*(G) \leq 10 \rightarrow \chi_c(G) \leq \rho^*(G) + 8 \leq 18, \\
(ii) & \rho^*(G) \leq 9 \rightarrow \chi_c(G) \leq \rho^*(G) + 7 \leq 16, \\
(iii) & \rho^*(G) \leq 8 \rightarrow \chi_c(G) \leq \rho^*(G) + 6 \leq 14, \text{ and} \\
(iv) & \rho^*(G) \leq 7 \rightarrow \chi_c(G) \leq \rho^*(G) + 6 \leq 13.
\end{array}
$$

PROOF. One simply modifies the proof of Theorem 3.1 by replacing the induction hypothesis of that theorem by the sharper bound listed in each case in the statement of the present theorem.

That these bounds cannot be sharpened by this inductive method, is illustrated by extremal face configurations $(3,10,10)$ for case (i), $(3,9,9)$ for case (ii), $(3,8,8)$ and $(4,7,8)$ for case (iii), and $(4,7,7)$ and $(5,6,7)$ for case (iv). ∎

Note that if we try to push the approach used in the proof of Theorem 3.2 further to include cases where $\rho^*(G) \leq 6$, the bounds we get in each case turn out to be just the same as those given by Ore and Plummer (1969), namely $2\rho^*(G)$. But the bound of Borodin (1984, Theorem 2; 1985, Theorem 3) is already one better in each of these cases. In fact, Borodin has also shown (1984, Theorem 1) that if $\rho^*(G) = 4$, then $\chi_c(G) \leq 6$. Finally, if $\rho^*(G) = 3$, then $\chi_c(G) \leq 4 = \rho^*(G) + 1$, by the Four Color Theorem. (See Appel and Haken (1977) and Appel, Haken and Koch (1977).)

In contradistinction to Theorem 3.2, we finally show that if $\rho^*(G)$ is sufficiently *large*, then we can also improve the bound of Theorem 3.1.

3.3. THEOREM. *Let G be a 3-connected plane graph. Then:*

$$(i) \qquad \rho^*(G) \geq 14 \rightarrow \chi_c(G) \leq \rho^*(G) + 8,$$
$$(ii) \qquad \rho^*(G) \geq 15 \rightarrow \chi_c(G) \leq \rho^*(G) + 7,$$
$$(iii) \qquad \rho^*(G) \geq 18 \rightarrow \chi_c(G) \leq \rho^*(G) + 6,$$
$$(iv) \qquad \rho^*(G) \geq 24 \rightarrow \chi_c(G) \leq \rho^*(G) + 5, \; and$$
$$(v) \qquad \rho^*(G) \geq 42 \rightarrow \chi_c(G) \leq \rho^*(G) + 4.$$

PROOF.   The proof is again by induction on $p = |V(G)|$.

(i). First suppose that $\rho^*(G) \geq 14$. Then the smallest applicable value for p is 14. But then every point can be colored differently using only $14 \leq \rho^*(G) < \rho^*(G) + 8$ colors.

Now suppose that $p \geq 15$ and that the conclusion is true for all 3-connected plane graphs with fewer than p points. Let G be a 3-connected plane graph with p points. Let v be a control point in G. (Recall that by definition of control point and by Lemma 2.2, $\deg v \leq 5$.)

First suppose that v is incident with a contractible line e. There are two cases to consider.

(a). Suppose that $\rho^*(G \circ e) = \rho^*(G)$. Then $\rho^*(G \circ e) \geq 14$, and by the induction hypothesis we can color $G \circ e$ in no more than $\rho^*(G \circ e) + 8 = \rho^*(G) + 8$ colors. Now checking all solution configuration possibilities at control point v present in the list in Section 2, we see that in all cases but one, $A \leq x + 7$. The one exception is the case $(3, 11, x)$, where $x = 11, 12$ or 13 and $A = x + 8$. But since $\rho^*(G) \geq 14$, in this case $x < \rho^*(G)$. Thus the coloring of $G \circ e$ in $\rho^*(G) + 8$ colors can be extended to a valid cyclic coloration of G in $\rho^*(G) + 8$ colors, as claimed.

(b). Therefore we may assume that $\rho^*(G \circ e) = \rho^*(G) - 1$. Then the induction hypothesis does not apply, but by Theorem 3.1 we can color $G \circ e$ in $\leq \rho^*(G \circ e) + 9 = \rho^*(G) + 8$ colors. But then proceeding as in Case (a), a cyclic coloration of G in $\rho^*(G) + 8$ colors is obtained.

So now suppose that there are no contractible lines at control point v. We will mimic the corresponding part of the proof of Theorem 3.1 at this point, contracting lines as in that proof. However, again we have two cases to consider.

(c). Suppose $\rho^*(G \circ e) = \rho^*(G)$. Thus $\rho^*(G \circ e) \geq 14$ and we may apply the induction hypothesis to color $G \circ e$ in no more than $\rho^*(G \circ e) + 8 = \rho^*(G) + 8$ colors. By inspecting the proof of Theorem 3.1, we find that a cyclic coloration of G in $\rho^*(G) + 8$ colors can be obtained.

(d). So suppose that $\rho^*(G \circ e) = \rho^*(G) - 1$. But then as in Case (b) above, we can obtain the desired cyclic coloration in $\rho^*(G) + 8$ colors of $G \circ e$, and hence of G, as in Case (c). This completes the proof of part

(i) of the theorem.

(ii). Next suppose that $\rho^*(G) \geq 15$. The proof for the smallest applicable value of p ($p = 15$) is again trivial.

Proceeding as in the proof of (i), if e is a contractible line as in any of the cases of part (i) and $\rho^*(G \circ e) = \rho^*(G)$, we can apply the induction hypothesis and complete the proof as in case (i).

If $\rho(G \circ e) = \rho^*(G) - 1$, we proceed as in part (i), except where we invoke Theorem 3.1 there, here we invoke part (i) of the present theorem.

The proofs of parts (iii), (iv) and (v) are handled similarly. ∎

In the case of Theorem 3.3, we note that we cannot immediately push this inductive method of proof further to obtain proofs that if the lower bound on $\rho^*(G)$ is even larger, then we can reduce the upper bound on $\chi_c(G)$ even more. The reason becomes apparent when we peruse the list of solution configurations of Section 2 once more. We see there that any attempt to go down to $\chi_c(G) \leq \rho^*(G) + 3$ is blocked by solution configurations $(3, 6, x)$, $(4, 4, x)$ and $(3, 3, 3, x)$, because in these three configurations the possible values for x are *not* bounded above.

4. Concluding remarks

We do not claim that the bounds we have obtained in the theorems of Section 3 are best possible. The triangular prism R_3 has 6 points, $\rho^*(R_3) = 4$ and $\chi_c(R_3) = 6$. So for this 3-connected planar graph, we have $\chi_c = \rho^* + 2$. The authors are indebted to one of the referees for pointing out that R_3 is only one of an infinite family of 3-connected planar graphs for which $\chi_c = \rho^* + 2$. This family is shown in Figure 4.1. In fact, the authors conjecture that $\chi_c \leq \rho^* + 2$ for all 3-connected plane graphs.

A second infinite family of graphs shown in Figure 4.2 (and due to this same referee) shows that we cannot weaken the hypothesis of Theorem 3.1 by replacing 3-connectivity with the condition that mindeg $G \geq 3$. In fact, there exists no constant k in this case such that $\chi_c \leq \rho^* + k$.

The triangular prism R_3 with the three lines joining the two triangles replaced by disjoint paths of equal lengths gives an infinite family of graphs with $\chi_c = \frac{3}{2}\rho^*$. It seems not to be known if the theorem of Ore and Plummer (1969) can be improved to obtain $\chi_c(G, \phi) \leq \frac{3}{2}\rho^*(G, \phi)$ for *all* 2-connected plane graphs (G, ϕ).

A non-planar analogue to the problem studied in this paper can be considered for general 3-connected graphs by the result of Tutte (1963) mentioned in the Introduction. In other words, one might color points

FIGURE 4.1.

FIGURE 4.2.

in such a way that each non-separating induced cycle has no two points with the same color. However, the theory of Euler contributions is no longer available as a tool.

References

K. ANDO, H. ENOMOTO AND A. SAITO

1985. Contractible edges in 3-connected graphs, Dept. of Information
Science, University of Tokyo, Technical Report 85-03, 1985.

K. I. APPEL AND W. HAKEN

1977. Every planar map is four colorable. I. Discharging, *Illinois J. Math.*
21, 1977, 429–490.

K. I. APPEL, W. HAKEN AND J. KOCH

1977. Every planar map is four colorable. II. Reducibility, *Illinois J. Math.*
21, 1977, 491–567.

O. V. BORODIN

1984. Solution of Ringel's problem about vertex bound coloring of planar
graphs and coloring of 1-planar graphs, *Metody Diskret. Analiz.* **41**,
1984, 12–26. (Russian).

1985. Consistent coloring of 1-imbeddable graphs, *Graphen und Netzwerke
— Theorie und Anwendungen (30. Intern. Wiss. Koll. TH Ilmenau 1985)*,
Ilmenau, D. D. R., 1985, 19–20. (Russian).

B. GRÜNBAUM

1967. *Convex Polytopes*, Wiley Interscience, London, 1967.

F. HARARY

1969. *Graph Theory*, Addison-Wesley, Reading, Mass., 1969.

H. LEBESGUE

1940. Quelques conséquences simples de la formule d'Euler, *J. de Math.* **9**,
1940, 27–43.

L. LOVÁSZ AND M. D. PLUMMER

1986. *Matching Theory*, Ann. Discrete Math., North-Holland, Amsterdam,
and Akadémiai Kiadó, Budapest, 1986..

O. ORE

1967. *The Four-Color Problem*, Academic Press, New York, 1967.

O. ORE AND M. D. PLUMMER

1969. Cyclic coloration of plane graphs, *Recent Progress in Combinatorics
(Proc. Third Waterloo Conf. on Combinatorics)*, Academic Press, New
York, 1969, 287–293.

M. D. PLUMMER

1985.   A theorem on matchings in the plane, *Conference in memory of Gabriel Dirac*, Ann. Discrete Math., North-Holland, Amsterdam, (to appear).

E. STEINITZ

1922.   Polyheder und Raumeinteilungen, *Enzykl. math. Wiss.* **3** *(Geometrie)*, Part 3AB **12**, 1922, 1–139.

C. THOMASSEN

1980.   Planarity and duality of finite and infinite graphs, *J. Combin. Theory Ser. B* **29**, 1980, 244–271.

A. T. WHITE

1973.   *Graphs, Groups and Surfaces*, North-Holland/American Elsevier, Amsterdam, 1973.

W. T. TUTTE

1963.   How to draw a graph, *Proc. London Math. Soc.* **13**, 1963, 743–768.

H. WHITNEY

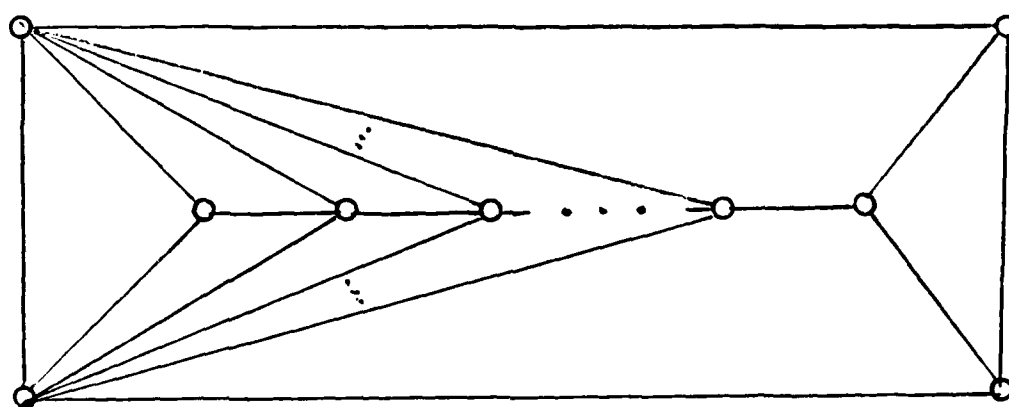1932.   Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54**, 1932, 150–168.
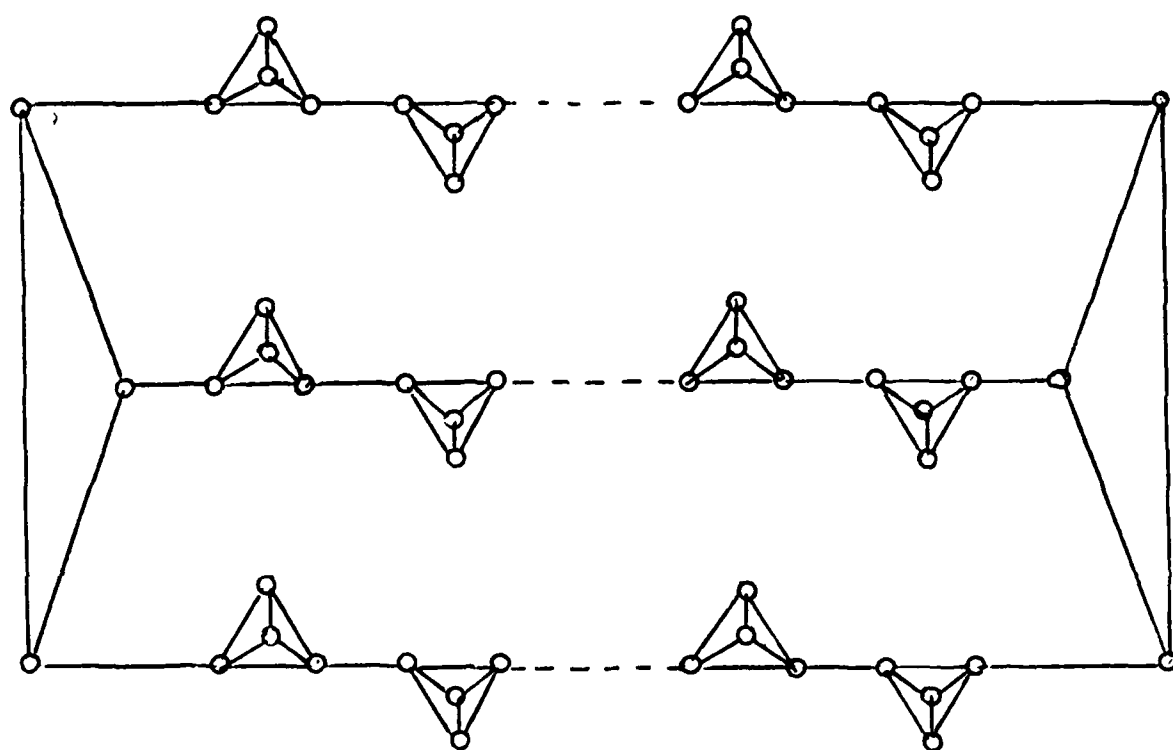
Figure 4.1

Figure 4.2

END
2-87
DTIC